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TURBULENT FLOW

By L. Prandtl

Lecture delivered before the International Congress
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N O T I C E

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TURBULENT FLOW.*

By L. Prandtl.

What I am about to say on the phenomena of turbulent flow is still far from conclusive. It concerns, rather, the first steps in a new path which I hope will be followed by many others.

The researches on the problem of turbulence which have been carried on in Göttingen for about five years, have unfortunately left the hope of a thorough understanding of turbulent flow very small. The photographic and kinetographic pictures have shown us only how hopelessly complicated this flow is. Figs. 1-4 are photographs of water flowing in a very long deep rectangular trough, as taken from above with a camera carried along on a car in the direction of the flow. The pictures differ greatly according to the speed of the car, but they are all unpleasantly complicated. I will show motion pictures of the same motions at the close of my lecture. Such pictures have hitherto been utilized only for statistical information on the magnitude of the velocity variations. Otherwise, we have not been able to learn much from them. That which I would call the

*"Ueber die ausgebildete Turbulenz," a lecture delivered at Zurich in September, 1926, before the International Congress for Applied Mechanics.

"great problem of developed turbulence," involving a thorough understanding and a quantitative calculation of the processes through which new phenomena are constantly being developed and also involving the determination of that mixing force which is produced in each individual instance by the contest of new development and damping, is not likely therefore to be soon solved.

It is always possible, however, if we forego a deeper understanding of the phenomena of turbulence, to follow theoretically various phenomena in a logical way controlled by experiments, especially regarding the mean motion in a given turbulent flow. Even the determination of the mean velocity, as a function of the place, is a technically very important task. The first step in this direction may be so characterized that the apparent viscosity forces, produced by the mixing, will be represented in such a form that they can be introduced into the hydrodynamic differential equations and thus furnish differential equations for the mean motion of turbulent flows.

Boussinesq has already undertaken this task. He originated the formula, now much used, which includes an "equivalent" A,* analogous to the viscosity whereby, in addition to the molecular tension $\tau = \mu \frac{\partial u}{\partial y}$ etc., we have the apparent molecular ten-

*I took this designation from the interesting and instructive article "Der Massenaustausch in freier Luft und verwandte Erscheinungen," by Professor Wilhelm Schmidt, Vienna (Hamburg, 1925). Boussinesq and subsequently other hydraulic engineers designated this quantity by ϵ.

sion

$$\tau = A \frac{\partial \bar{u}}{\partial y} \text{ etc.} \tag{1}$$

in which $\bar{u}$ = the periodical mean value of the velocity component u. This formula has the disadvantage, however, that the equivalent itself depends in turn on the velocity, which must therefore first be found.

After many futile attempts, I finally succeeded in obtaining an expression for the apparent friction which, although only a rough approximation, still quite well represents the essential flow characteristics of slightly viscous fluids and is therefore free from the abovementioned fault.

In order to formulate the exchange of momentum, which produces the apparent friction, it is customary, according to the method of O. Reynolds, to divide the momentary velocity into two components, the periodical mean value and the oscillation about this value.

$$u = \bar{u} + u' \qquad v = \bar{v} + v' \qquad w = \bar{w} + w'$$

In the mean value the oscillations now give an apparent state of tension through the components

$$\rho \overline{u'^2}, \qquad \rho \overline{u' v'}, \quad \text{etc.}$$

in which ρ = density, overlined as sign of the mean value formation. These expressions are well known, and the task is now to obtain for them a form in which the mean flow ("fundamental flow") $\bar{u}\ \bar{v}\ \bar{w}$ occurs.

At this point it is essential to introduce the characteristic length for the turbulent condition, which here plays a similar role to that of the free wave length in the kinetic gas theory. It can be considered as the diameter of the simultaneously moved fluid masses, but also as the distance traversed by such a fluid mass before it loses its individuality by mixing with neighboring masses. It is easily found that these two distances, if the Reynolds Number* is large enough, differ only through a constant factor (work of overcoming resistance in penetrating foreign fluid masses = kinetic energy of the mass). Following the second definition, we will designate this length as the "mixing path" (or "mixing distance") and represent it by l. If we assume that a moving fluid mass, situated in a current with velocity decrease crosswise to the current, possesses a velocity equal to the mean velocity at the point where it originated, and that it is shifted by the mixing distance l transversely to the current, its velocity will then differ from the mean velocity at the new point, and this velocity, in its first approximation, equals $l \frac{\partial \bar{u}}{\partial y}$, when the mean direction of flow relative to the X axis is chosen. The mean oscillation u' can therefore be put proportional to $l \frac{\partial \bar{u}}{\partial y}$. The transverse motion v' can be considered as being produced in the manner that two fluid masses with different u', which find themselves in one another's presence, either come together or move farther apart. The velocities v' thus

*That is, the Reynolds Number obtained from the diameter of the fluid mass and from the relative velocity.

produced can therefore be put proportional to u'. The apparent molecular tension $(\tau' = \rho \, \overline{u' v'})$ therefore becomes

$$\tau' = \rho \, l^2 \left(\frac{\partial \overline{u}}{\partial y}\right)^2,$$

when the proportionality factors of u' and v' and also the correlation factor, which would come into the formation of the mean product, are expressed. If the sign, which must change with that of $\frac{\partial \overline{u}}{\partial y}$, is taken into consideration, it is more correctly written

$$\tau' = \rho \, l^2 \left|\frac{\partial u}{\partial y}\right| \frac{\partial u}{\partial y} \tag{2}$$

Comparison with the Boussinesq form shows that agreement will be obtained when the "equivalent" is written

$$A = \rho \, l^2 \left|\frac{\partial u}{\partial y}\right| \tag{3}$$

This simple formula has given very good results, in spite of its obvious imperfections, for the case used as the basis of its derivation, namely, a flow with velocity decrease transverse to the current. Above all, in agreement with hydraulic experience, it gives resistances proportional to the square of the velocity, as can be easily seen from its structure. We must bear in mind, however, in the results which are obtained with this formula, that it gives only a first rough approximation. The solutions of flow problems with this viscosity factor show somewhat unusual characteristics, when the customary viscosity factor is expressed as numerically small. Instead of an asymptotic transition to constant velocity, a place is found somewhere in the finite, where the finite sharp curve tangentially joins a horizontal straight line. Velocity maxima always have such a form, that the radius of curvature there drops to zero. It is in the vicinity of the maximum $\overline{u}_{max} - \overline{u}_{prop}$. $|y - y\,l|^{3/2}$. This behavior is connected with the condition that for $\frac{\partial \overline{u}}{\partial y} = 0$ the equivalent becomes zero according to our formula. In reality this is not exactly what happens, since the equivalent does not entirely disappear, due to the disturbance in the adjacent regions. If, for the refinement of the theory, an extension of the equivalent is assumed, there is then obtained, at the places $\frac{\partial \overline{u}}{\partial y} = 0$, still another equivalent differing from zero, and correct asymptotes and maxima with finite curvature are again obtained. Nevertheless, many experiments show that the curvature of the velocity curve at the place of the maxima is especially large, from which it is to be concluded that the equivalent, although not zero, is still considerably smaller than in the vicinity, so that the simple formula 2 cannot be denied a certain intrinsic justification.

Thus far we have considered only the components of the molecular tension, which usually play the principal role in hydraulic problems. It is not difficult, however, even for somewhat more general cases to develop an expression compatible with the symmetry characteristics, which furnishes a full tension tensor. For example, the factor $\left|\frac{\partial \overline{u}}{\partial y}\right|$ in formula 2 can be replaced by the maximum velocity decrease transverse to the streamline, and the second factor by $\frac{\partial \overline{u}}{\partial y}$ by the tensor $\Delta \overline{u} + \overline{u} \Delta$. Certain experiments, to which I will recur, indicate, however, that the matter is not quite so simple. The equivalent itself must rather be regarded as a tensor, so that in the generalization of formula 2, as soon as we pass beyond the planar problem, we have the product of two tensors.

The "mixing path" l requires further special investigation. It may generally be considered as a Φ function of the place, concerning which it is first to be established, that it must become zero on approaching a wall, because the transverse motions are here prevented. The observations in smooth-walled tubes indicate, however, that the viscosity still has a slight effect. In the field of the Blasius resistance formula l must be put proportional with

$$y^{6/7}\left(\frac{\nu}{\sqrt{\tau_{boundary}\ \rho}}\right)^{1/7}$$

in order, by using formula 2, to obtain the correct relation of the resistance to Reynolds Number and simultaneously the proportionality of the speed with the seventh root of the wall resistance y.*

The relations appear to be simpler in those turbulent motions in which no walls participate as, e.g., in the mingling

*Proportionality with $\frac{l u}{b}$ follows from $v' \sim l \frac{\partial \overline{u}}{\partial y}$ for a mean value of v' (b = width, $\overline{u}$ = mean value).

of flowing fluid with the surrounding non-flowing fluid and in the retardation of the flow behind a moving object. In these cases of "free turbulence," it can be assumed, at least for sufficiently large Reynolds Numbers that, in comparable cases, the phenomena are always geometrically and mechanically similar in a cross section transverse to the longitudinal direction. Consequently the mixing distances, with increasing width of the stream or wake, are always proportional to the width of the stream, whereby the consideration at the base of formula 2 also enables the transverse velocities to be proportional to the mean relative velocity $\bar{u}$ with respect to the undisturbed fluid. This in itself plausible assumption concerning l may moreover be derived, if so desired, as a result of the fact that the retardation of the mean motion $\bar{u}$ is produced by the molecular tensions given in formula 2, and that, on the other hand, the periodical increase in the width of a section of the turbulent flow, etc., is proportional to the velocity of the transverse motions. The results of the various ways of considering the subject therefore agree well with one another.

With the addition of the law of inertia for the principal motion, the ratio for the increase in the width and for the decrease in velocity with increasing distance from the place of disturbance can be forecast for numerous instances. In the widening of the flow, there is already developed, without the law of inertia (or momentum), an increase of the width proportional to the distance from the hole* and consequently, due to the law of inertia, a velocity decrease inversely proportional to the first power of the distance for a flow of circular cross section or inversely proportional to the (square) root of the distance in the case of a flow coming out of a slot.

In the wake of a rod or rotational body placed transversely to the flow, the width increases proportionally to the square root or cubic root of the distance, while the velocity decreases in inverse proportion to the square root or the 2/3 power of the distance. In all these rules it is, moreover, assumed that the velocities, or the deviations of the velocity from the undisturbed flow, are small in comparison with those of the place of disturbance.

The spreading of a stream with circular cross section and the wake of a rotational body may be taken as examples. Let b represent the width of the stream or wake. Let $l = \alpha b$, so that the transverse velocity $v' = l \frac{\partial u}{\partial y} \sim \frac{l u}{b} \sim \alpha u$.

From the expression

$$\frac{Db}{dt} \sim v' \qquad (a)$$

is obtained, by estimating the stream

$$u \frac{db}{dx} \sim \alpha u$$

hence $b \sim \alpha x$.

*This result can also be obtained through a similarity calculation.

The constant momentum in all the cross sections becomes

$J \sim u^2\ b^2$, whence $u = \frac{\text{const.}}{x}$

For the wake $\frac{Db}{dt} = U\ \frac{db}{dx}$ where U is the velocity of the undisturbed flow and hence

$$U\ \frac{db}{dx} \sim \alpha\ u \tag{b}$$

The momentum is $J \sim \rho\ U\ u\ b^2$, which equals the resistance (or drag) $W = c_W\ \frac{\rho\ U^2}{2}\ F$, in which F = cross section of body and c_W = coefficient of drag. $J = W$ gives

$$u \sim \frac{c_W\ U\ F}{2\ b^2} \tag{c}$$

which, when introduced into formula 1, becomes

$$b^2\ \frac{db}{dx} \sim c_W\ F,$$

hence

$$b \sim \sqrt[3]{c_W\ F\ x}$$

and

$$u \sim U \sqrt[3]{\frac{c_W\ F}{x^2}}$$

The experimental proof of the rules thus obtained, in so far as they relate to the widening of the stream, has already been given. In the wake flows, the preliminary experiments show certain deviations, which may possibly be connected with too small Reynolds Numbers. The final conclusion has not yet been reached.

If, as mentioned, we assume the mixing distance l in formula 2 to be proportional to the stream width determined by the above-mentioned rules and also assume that it is constant over the whole width at one and the same distance, we then have sufficient data to solve the hydrodynamic differential equations, supplemented by the friction member of formula 2, in the same manner as customary in boundary layer calculations. (The pressure differences transverse to the direction of flow are disregarded, as likewise the effects of other deformation members except $\frac{\partial \overline{u}}{\partial y}$.) On this basis my fellow worker, Dr. Tollmien, has made various calculations, which will appear shortly in "Zeitschrift für angewandte Mathematik und Mechanik." The accompanying diagrams show some of the results. In this connection, the mixing, with the neighboring air, of a broad uniform air flow, which comes from an opening, is treated as a special previously unobserved form of flow. This case may be a little more closely investigated as an example. Here also, as in the other flow extension problems, l is to be put proportional to x (distance from opening). A dependence on y is not to be assumed, so that we can put $l = cx$. Here the formula is $\overline{u} = f(\eta)$, in which $\eta = y/x$. On one boundary of the field $\overline{u} = U$; on the other boundary $\overline{u} = 0$ (Fig. 5). We now have

$$\overline{u}\,\frac{\partial \overline{u}}{\partial x} + \overline{v}\,\frac{\partial \overline{u}}{\partial y} = \frac{1}{\rho}\,\frac{\partial \tau}{\partial y} = 2\ cx^2 \left|\frac{\partial \overline{u}}{\partial y}\right| \frac{\partial^2 \overline{u}}{\partial y^2}\,.$$

By introducing the flow function $\psi = x\,F(\eta)$, where $F(\eta) = \int f(\eta)\, d\,\eta$, we obtain, after a short calculation, the very simple differential equation

$$F F'' + 2 c^2 F'' F''' = 0$$

which can be solved both through $F'' = 0$ (u = const.) and also through

$$F + 2 c^2 F''' = 0$$

The latter differential equation represents a sort of damped oscillation, of which a half-oscillation represents the function to be considered for the mixing region. This region is joined tangentially on one side by $u = U$ and on the other side by $u = 0$ (Fig. 6). In other cases, similar differential equations are obtained, most of which, however, are more difficult to solve.

After the solution, the pressure differences transverse to the flow direction, which were at first disregarded, can be calculated from the transverse accelerations and the apparent transverse tensions. In the hitherto verified examples, these pressure differences were found to be less than 1% of the dynamic pressure at the maximum velocity in the given cross section, so that their neglect is justified. Everywhere in the stream there is found a slight positive pressure. In the air streams of the aerodynamic laboratories, this positive pressure has a technical importance for precision experiments, e.g., for the calibration of pressure gauges for air-speed measurements. For the large Göttingen air stream, whose mingling with the surrounding air has been measured, Dr. Tollmien calculated a positive pressure of about $0.006 \frac{\rho U^2}{2}$. Perhaps the following figures will also be of interest: width of mixing zone $b = 0.255\ x$; mixing distance $l = 0.0174\ x = 0.068\ b$; inflow velocity of the air (from a state of rest) into the air stream to replace the air carried away by the latter, $\overline{v} = 0.032\ U$. The agreement of the calculated with the observed curves is well shown by Fig. 8, where two pressure records are given, with the theoretical curves represented by added dash lines.

In addition to the above-mentioned cases, there have already been calculated the lessening of the turbulence behind a screen of parallel wires, whereby the velocity deviations diminish in an inverse ratio to the distance, and the periodical increase in the turbulent layer, which proceeds from a separation layer with velocity increase. The increase $\frac{db}{dt}$ occurs with a constant velocity proportional to the strength of the velocity increase. The mixing distance $l = c\ (u_1 - u_2)\ t$. (The velocity curve is here given according to formula 2 simply by a function of the third degree of the form $Ay - By^3$.)

A theoretical determination of the hitherto only empirically determinable magnitude c can be obtained, I believe, in this as in other suitable examples, from the claim that the layer thus developed is no longer unstable toward small oscillations. Of course it is not easy actually to calculate the oscillations of such a layer with allowance for the apparent friction, even of the oscillation motion. The proof of this claim is that a too small c produces too small transition layers or streams, which are then dynamically unstable. The instability then indicates the formation of vortices, i.e., increased mixing, widening, etc., q.e.d. (It may also be that instability is developed for a short time in the disturbance of a given wave length or oscillation period, which is then, however, replaced by stability. In such cases the c value actually involved will depend on the initial disturbances.)

We must now discuss another question, which concerns the more complex problems, which cannot be so readily subjected to theoretical treatment. Here formula 2 or its extensions can be employed to determine the magnitude of the mixing distance in dependence on the location in the fluid, for the flow established by the experiment. This furnishes an especially clear picture of the intensity of the mixing phenomena at each individual point, and we receive thereby, since it concerns a simple length in the mixing path, a very easily presentable measure of this intensity, which can be readily transferred from the model to the full-scale object. Thereby remarkably small differences in the mean value of l are manifested in the previously tested cases. In smooth troughs of uniform cross section, as in the case of an increasing or decreasing cross section and as in the violent phenomena behind an obstacle and in the widening of streams, mixing paths are developed, which constitute 1/8 to 1/10 of the depth of the water or 1/2 of the width of the channel or half of the actual width of the stream. Figs. 9-10 give the l curves for smooth widened and narrowed channels. Rough channels are being tested in Göttingen, but the results have not yet been published.

One of the most important works of the near future will involve the study of the friction layers on solid bodies and especially the conditions for the harmful separation of the flow from airplane wings, the walls of diffusers (exit cones of wind tunnels), etc. Here also beginnings have already been made.

In concluding, I will speak of one more group of phenomena which relates to the spatial turbulence problem (in contrast to the periodical or symmetric rotational problem, which alone has thus far been discussed). It concerns the velocity distribution in other than cylindrical tubes. I have had especially careful measurements made in this connection, the results of which were very surprising. Instead of a distribution with ever more rounded "Isotachen" (lines of like velocity), as obtained in laminar flow, there were obtained for triangular and rectangular channels the lines shown in Figs. 11-12, of which those for the rectangular channel are still more peculiar than for the triangular one. An article on old observations of the spiral motion of water in a straight channel ("Die Wasserkraftlaboratorien Europas" Berlin, 1926, pp. 66-67), furnished me the basis for a usable explanation. The water develops "secondary motions" in all channels of uniform but not circular cross section in such manner that in a corner the fluid along the middle of the angle flows into the corner and on both sides of the middle it flows out of the corner. These flows, in connection with the usual turbulent mixing path, enable the explanation of the observed phenomena. Momentum is ever communicated to the corners, thus producing the great velocities there. Fig. 13 shows the secondary flows for the triangular and rectangular channels. It is seen how the inward flow from the wall develops regions of subnormal velocity at the ends of the long sides and also in the middle of the short sides. For the confirmation of the new views, I recently had measurements made with another tube, as shown in Fig. 14. The results, which otherwise would be very surprising, confirm the explanation in the best manner. The currents coming from the corners develop, in the middle, a double eddy, which carries the water in the middle line toward the level fluid and carries it away again at the projecting corners, thus producing hypernormal velocity in the middle and subnormal velocity on the sides. Such secondary currents are generated on the surface of open troughs, as shown by experiments. The free surface is accordingly no cross section through an even flow. Moreover, the photographically determined velocity distribution on the surface agrees well with that found with a Pitot tube.

How then are the secondary currents to be explained? In my opinion there is no other explanation than this. The mixing motion is of such a nature that, alongside the to-and-fro motion in the direction of the strongest velocity gradient, there is a still stronger oscillation, crosswise to the former and hence in the direction of the "Isotachen." If this is true, it is explained by a simple momentum consideration that, through this kind of motion, forces are generated on the convex side of the "Isotache," whose strength is proportional to the sharpness of the curvature. Consideration of the mean values of $(u')^2$, $u' v'$, and $(v')^2$ give results which agree with this momentum consideration.

As to why the mixing motion is of this nature, that is a question which belongs to the previously mentioned "great turbulence problem," to which unfortunately I do not have the answer. In any event, this phenomenon plainly indicates that the developed turbulence is an essentially three-dimensional motion. This circumstance seems, however, to postpone the solution of the great turbulence problem to the very distant future, because our present mathematical resources are quite insufficient for three-dimensional fluid motions. The way indicated in this lecture will, however, suffice in general for the arranging of the experimental results, so that we can dispense with a complete explanation.

Translation by Dwight M. Miner,
National Advisory Committee for Aeronautics.

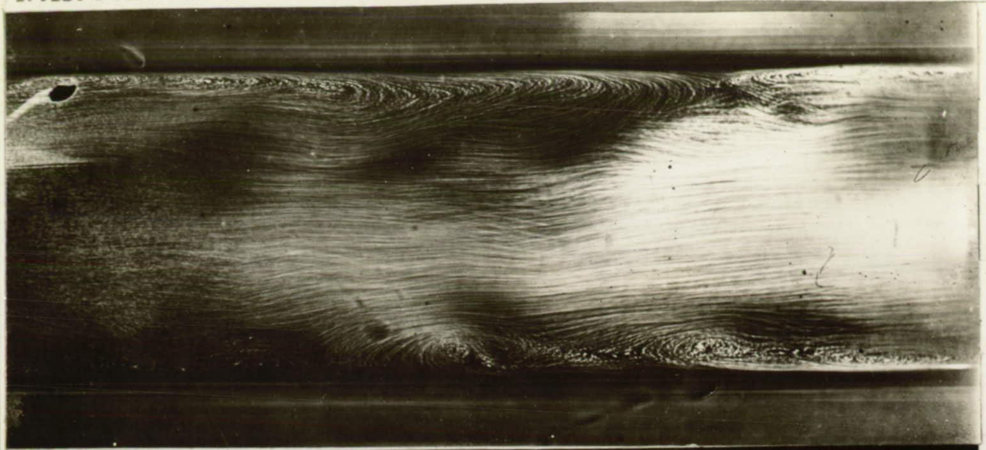

Fig.1

U = 6 cm/s
(.197 ft./sec.)

Fig.2

U = 7 cm/s
(.230 ft./sec.)

Fig.3

U = 8 cm/s
(.262 ft./sec.)

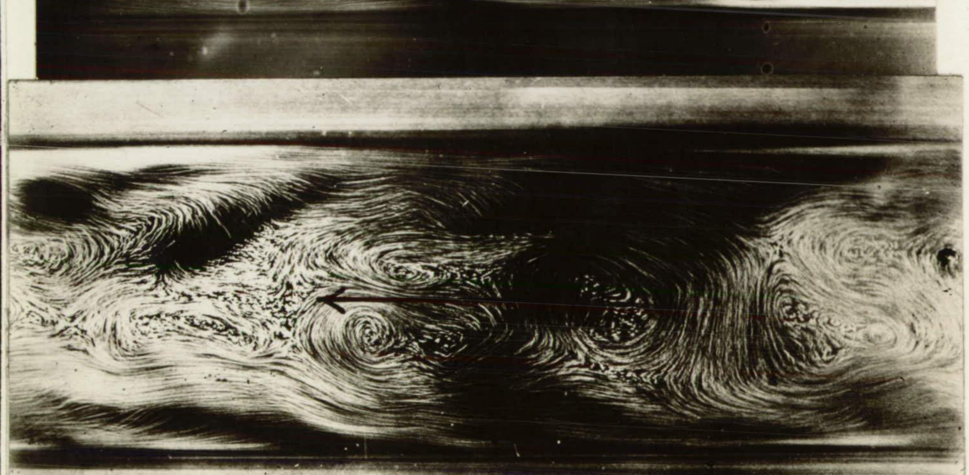

Fig.4

U = 9 cm/s
(.295 ft./sec.)

3015 A.S.

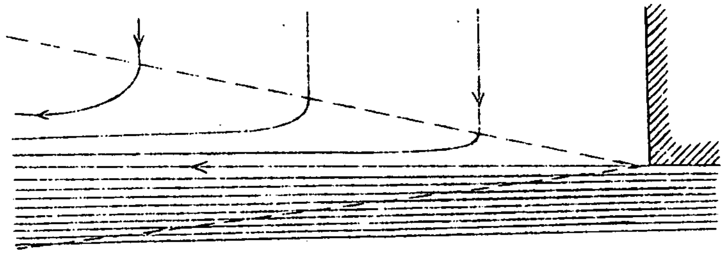

Fig.5

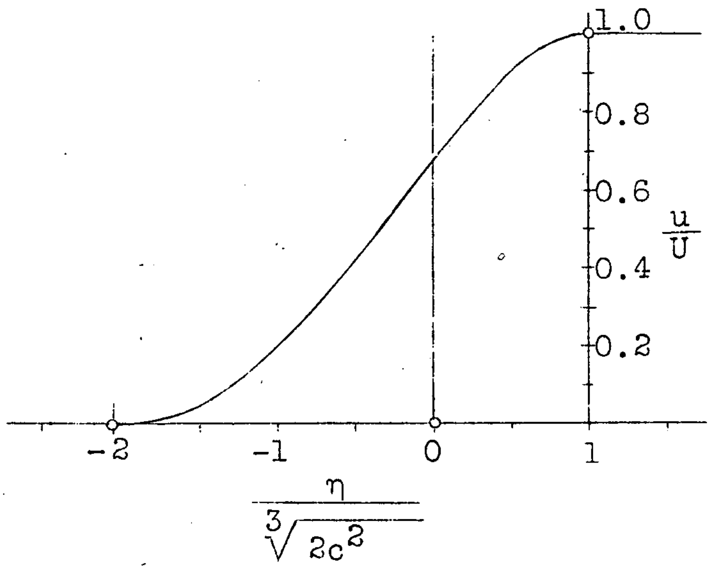

Fig.6

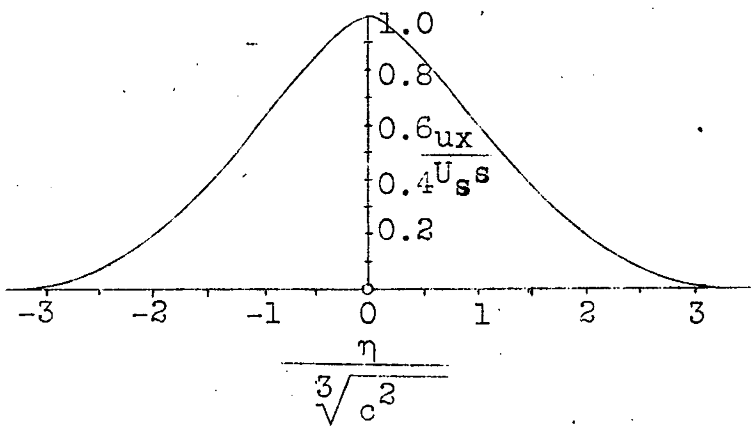

Fig.7

Spreading of a stream.

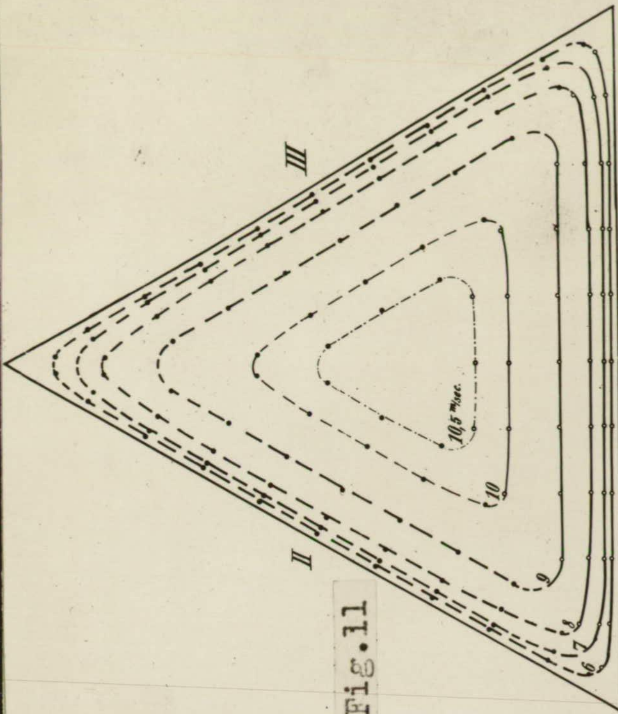

Fig.11

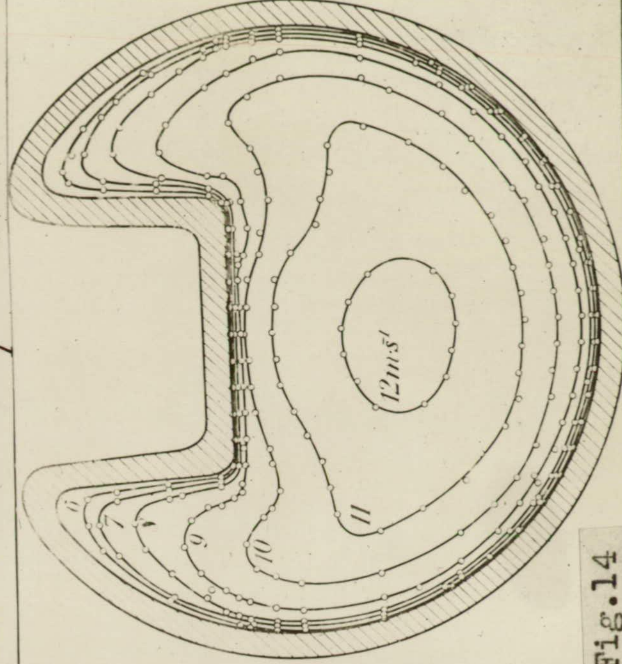

Fig.14

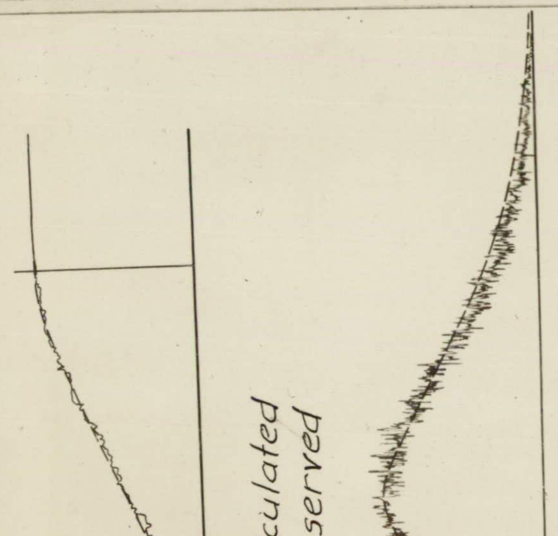

Fig.8

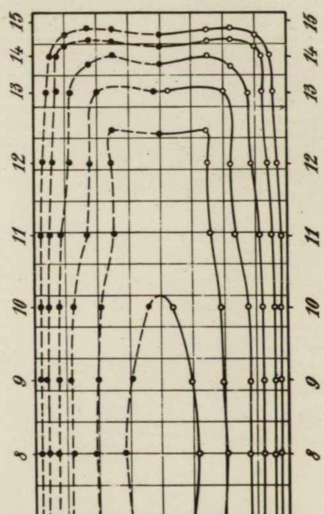

Fig.12

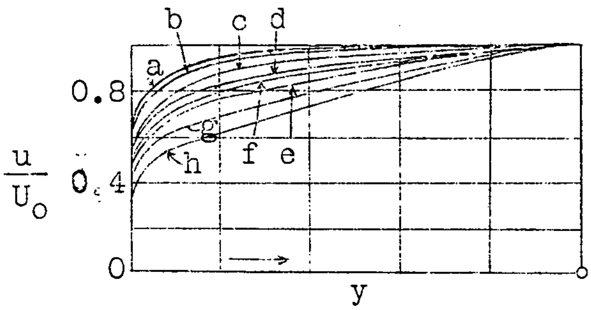

Fig.9 Velocity distribution.

a = -3.0° e = 1.6°
b = -2.2° f = 0.8°
c = -1.3° g = 2.3°
d = 0° h = 2.9°

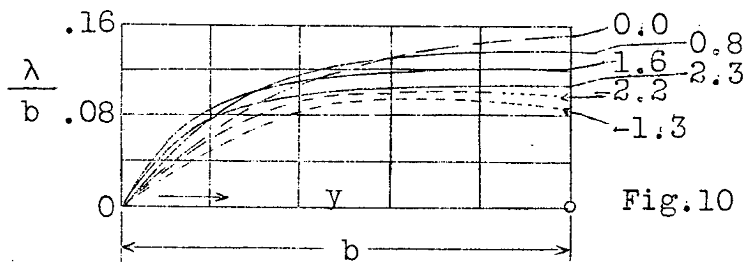

Fig.10 Distribution of l

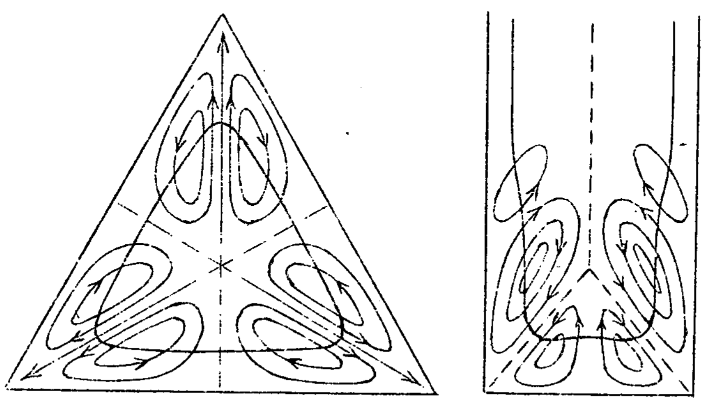

Fig.13.